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LETTER TO THE EDITOR

Spontaneous and stimulated emission from atoms prepared in the super-radiant state

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Abstract. Spontaneous and stimulated emission from atoms prepared in the super-radiant state are studied, in the non-resonant case. An exact equation of motion for R_3 and approximate solutions are obtained.

Super-radiant emission for two-level atoms in the resonant case has been studied by various authors, in the past, at various approximations (Bonifacio and Preparata 1970, Bonifacio *et al* 1971, Glauber and Haake 1976).

In this letter, we make use of the basic non-resonant model for super-radiance to derive the equation of motion for R_3, calculate $\langle R_3\rangle(t)$ and $\bar{n}(t)$ in the cases of spontaneous and stimulated emission and determine the photon statistics for short times ($\tau \ll r^{-1/2}$).

The basic Hamiltonian is:

$$H = \hbar\omega a^\dagger a + \hbar\omega_0 R_3 + \hbar K(aR^+ + a^\dagger R^-). \tag{1}$$

Using Heisenberg's equation of motion for R_3, we write:

$$\dot{R}_3 = -\frac{\mathrm{i}}{\hbar}[R_3, H] = (\mathrm{i}K)(a^\dagger R^- - aR^+), \tag{2}$$

and

$$\ddot{R}_3 = -\frac{\mathrm{i}}{\hbar}[\dot{R}_3, H]. \tag{3}$$

After some straightforward algebra, one gets:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \hat{\Omega}^2\right)R_3 + 2K^2(R^2 - 3R_3^2) = \Delta(\omega^{\mathrm{RWA}} - \omega\hat{N}), \tag{4}$$

where

$$\Delta = \omega_0 - \omega, \qquad \hat{\Omega}^2 = (2K^2)(2\hat{N}+1) + \Delta^2, \qquad \hat{N} = a^\dagger a + R_3, \tag{5}$$

$\omega^{\mathrm{RWA}} = H/\hbar$ (the Hamiltonian in the rotating wave approximation). Scaling the time $\tau = Kt$, equation (4) can be written as:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \hat{\Omega}_1^2\right)R_3 - 6R_3^2 = \Delta_1(\omega_1^{\mathrm{RWA}} - \omega_1\hat{N}) - 2R^2, \tag{6}$$

where

$$\Delta_1 = \Delta/K, \qquad \hat{\Omega}_1^2 = 2(2\hat{N}+1) + (\Delta/K)^2,$$
$$\omega_1^{\mathrm{RWA}} = \omega^{\mathrm{RWA}}/K, \qquad \omega_1 = \omega/K. \tag{7}$$

Equation (6) has been solved exactly for one atom (see, for example Allen and Eberly 1975 or Jaynes and Cummings 1963). In this equation all operators involved besides R_3 are constants of motion. Specifically, $\hat{\Omega}_1^2$ and the whole second member are constants.

An approximate solution can be obtained by assuming:

$$\langle R_3^2\rangle = \langle R_3\rangle^2, \tag{8}$$

which amounts to neglecting the fluctuations of R_3. In the super-radiant case this should be a good approximation, since fluctuations occur around $m \sim 0$.

The initial state of the system is specified by:

$$|r, m=0, n(0), t=0\rangle.$$

Using the condition (8), equation (6) becomes a c-number differential equation. Define

$$y(\tau) = \langle R_3\rangle_{\substack{m=0\\ n(0)}}(\tau),$$

then

$$\ddot{y} + (\langle\hat{\Omega}_1^2\rangle_{\substack{m=0\\ n(0)}})y - 6y^2 = \langle\Delta_1(\omega_1^{\mathrm{RWA}} - \omega_1\hat{N}) - 2R^2\rangle_{\substack{m=0\\ n(0)}} \tag{9}$$

where $\langle\hat{\Omega}_1^2\rangle_{m=0,n(0)}$ and $\langle\Delta_1(\omega_1^{\mathrm{RWA}} - \omega_1\hat{N})\rangle_{m=0,n(0)}$ can be easily computed, since they are expectation values of conserved operators:

$$\langle\hat{\Omega}_1^2\rangle_{\substack{m=0\\ n(0)}} = 4n(0) + 2 + \Delta_1^2,$$
$$\langle\Delta_1(\omega_1^{\mathrm{RWA}} - \omega_1\hat{N}) - 2R^2\rangle_{\substack{m=0\\ n(0)}} = -2r(r+1). \tag{10}$$

Making use of equation (10), equation (9) becomes

$$\ddot{y} + y(4n(0) + 2 + \Delta_1^2) - 6y^2 = -2r(r+1). \tag{11}$$

Multiplying equation (11) by $\dot{y}$ and integrating, choosing zero for the integration constant, one gets

$$(\dot{y})^2 = 4y(y-y_1)(y-y_2), \tag{12}$$

where

$$y_{1,2} = (4n(0) + 2 + \Delta_1^2)/8 \pm [(4n(0) + 2 + \Delta_1^2)^2/64 + r(r+1)]^{1/2}. \tag{13}$$

The solution of equation (12) can be readily expressed in terms of elliptic functions:

$$y(\tau) = \langle R_3\rangle_{\substack{m=0\\ n(0)}}(\tau) = \frac{y_1 y_2}{y_1 - y_2} sd^2(u, m), \tag{14}$$

where:

$$u = \tau(y_1 - y_2)^{1/2}, \qquad m = -\frac{y_2}{y_1 - y_2} > 0. \tag{15}$$

For short times, $sd(u, m) \approx u$, and we get:

$$\langle R_3\rangle_{\substack{m=0\\ n(0)}}(\tau) = \left(\frac{y_1 y_2}{y_1 - y_2}\right) u^2 = -r(r+1)\tau^2, \tag{16}$$

which is essentially Dicke's result (Dicke 1954).

To compute $\bar{n}(\tau)$, we observe that

$$\bar{n}(\tau) + \langle R_3\rangle(\tau) = \text{constant} = \bar{n}(0),$$

so that

$$\bar{n}(\tau) = \bar{n}(0) - \frac{y_1 y_2}{y_1 - y_2} sd^2(u, m), \tag{17}$$

or, in a final form

$$\bar{n}(\tau) = \bar{n}(0) + [r(r+1)sd^2(u, m)]/2[(4n(0)+2+\Delta_1^2)^2/64 + r(r+1)]^{1/2}. \tag{18}$$

In the special case $\Delta_1 = 0$ (resonance), $n(0) = 0$ (spontaneous emission) and $r \gg 1$, equation (18) becomes:

$$\bar{n}(\tau) = \tfrac{1}{2} r sd^2(u, m), \qquad \text{with } u \approx \tau(2r+1)^{1/2} \quad \text{and } m \sim \tfrac{1}{2}. \tag{19}$$

These results agree with Bonifacio and Preparata (1970).

Finally, to study the photon statistics in the spontaneous emission case, consider the probability amplitude for n photons at time τ (scaled time), given by

$$p(n, \tau) = \langle m = -n|\langle n|\exp(-\mathrm{i}H\tau/\hbar K)|0\rangle|m = 0\rangle. \tag{20}$$

Differentiating both sides of equation (8) with respect to τ we get:

$$\mathrm{i}\dot{p}(n, \tau) = \langle m = -n|\langle n|[\omega_1 \hat{N} + \Delta_1 R_3 + (a^\dagger R^- + aR^+)] \exp(-\mathrm{i}H\tau/\hbar K)|0\rangle|0\rangle. \tag{21}$$

By using the well known properties of R_3, R^+, R^-, equation (21) becomes:

$$\begin{aligned}\mathrm{i}\dot{p}(n, \tau) = {} & [n(r-n+1)(r+n)]^{1/2} p(n-1, \tau) - n\Delta_1 p(n, \tau) \\ & + [(n+1)(r+n+1)(r-n)]^{1/2} p(n+1, \tau).\end{aligned} \tag{22}$$

Equation (22) cannot be solved exactly, but we can solve it in the short-time limit, if we make the following assumption:

$$n \ll r \qquad \text{or} \qquad \tau \ll r^{-1/2}.$$

In this case, the equation (22) becomes

$$\mathrm{i}\dot{p}(n, \tau) \approx r n^{1/2} p(n-1, \tau) + r(n+1)^{1/2} p(n+1, \tau) - n\Delta_1 p(n, \tau). \tag{23}$$

Equation (23) can be solved exactly; the solution for the n-photon probability amplitude is:

$$p(n, \tau) = (-1)^n \frac{(r\tau)^n}{(n!)^{1/2}} \exp\left[-\frac{r^2}{2}\left(\tau - \frac{\mathrm{i}\Delta_1}{r^2}\right)^2\right] \exp[\mathrm{i}(n-1)\Delta_1 \tau]. \tag{24}$$

Therefore, the n-photon probability turns out to be

$$|p(n, \tau)|^2 = \frac{(r^2\tau^2)^n}{n!} \exp(-r^2\tau^2) \exp[(\omega_0 - \omega)^2/2K^2 r^2]. \tag{25}$$

If we take a large number of atoms, consistent with our earlier assumption, then equation (25) is a Poisson distribution, the same as in the resonant case (Bonifacio and Preparata 1970) with $\bar{n} = r^2\tau^2$.

References

Allen L and Eberly J H 1975 *Optical Resonance and Two-Level Atoms* (New York: Wiley)
Bonifacio R and Preparata G 1970 *Phys. Rev.* A **2** 336
Bonifacio R, Schwendimann P and Haake F 1971 *Phys. Rev.* A **4** 854
Dicke R H 1954 *Phys. Rev.* **93** 99
Glauber R J and Haake F 1976 *Phys. Rev.* A **13** 357
Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89